Definite Integration with u-Substitution - Homework

Find the values of the following definite integrals. Verify using your calculator. Some will use *u*-substitution, others will not.

1. $\int_{-2}^{2}\left(x^3-1\right)dx$

$\left[\frac{x^4}{4}-x\right]_{-2}^{2}$

$4-2-(4+2)=-4$

2. $\int_{0}^{4}x\left(\sqrt{x}-1\right)dx$

$\left[\frac{2x^{5/2}}{5}-\frac{x^2}{2}\right]_0^4$

$\frac{64}{5}-8-0=\frac{24}{5}$

3. $\int_{0}^{\pi/3}\sin(2x)\,dx$

$\frac{1}{2}\int_0^{\pi/3}2\sin 2x\,dx \qquad u=2x, du=2\,dx$

$\frac{1}{2}\int_0^{2\pi/3}\sin u\,du \qquad x=0,u=0 \quad x=\pi/3,u=2\pi/3$

$\frac{1}{2}(-\cos u)\Big|_0^{2\pi/3}=\frac{1}{4}+\frac{1}{2}=\frac{3}{4}$

4. $\int_{0}^{\pi/12}\left(1-\cos 2x\right)dx$

$x\Big|_0^{\pi/12}-\frac{1}{2}\int_0^{\pi/12}2\cos 2x\,dx \mid u=2x, du=2x\,dx$

$\frac{\pi}{12}-\frac{1}{2}\int_0^{\pi/6}\cos u\,du \mid x=0,u=0 \mid x=\pi/12,u=\pi/6$

$\frac{\pi}{12}-\frac{1}{2}(\sin u)\Big|_0^{\pi/6}=\frac{\pi}{12}-\frac{1}{4}$

5. $\int_{0}^{1}2x\left(x^2+1\right)^2dx$

$u=x^2+1, du=2x\,dx$

$\int_0^1 2x\left(x^2+1\right)^2dx=\int_0^2 u^2du$

$x=0,u=1 \mid x=1,u=2$

$\left(\frac{u^3}{3}\right)\Big|_1^2=\frac{8}{3}-\frac{1}{1}=\frac{7}{3}$

6. $\int_{0}^{3}x\sqrt{9-x^2}\,dx$

$u=9-x^2, du=-2x\,dx$

$\frac{-1}{2}\int_0^1 -2x\left(9-x^2\right)^{1/2}dx$

$x=0,u=9 \mid x=3,u=0$

$\frac{-1}{2}\int_9^0 u^{1/2}du=\frac{1}{2}\left(\frac{2u^{3/2}}{3}\right)\Bigg|_0^9=9$

7. $\int_{0}^{5}\left|x-4\right|dx$

$\int_0^4(4-x)\,dx+\int_4^5(x-4)\,dx$

$\left[4x-\frac{x^2}{2}\right]_0^4+\left[\frac{x^2}{2}-4x\right]_4^5$

$16-8+\frac{25}{2}-20-(8-16)=\frac{17}{2}$

8. $\int_{0}^{4}\left|x-\sqrt{x}\right|dx$

$\int_0^1\left(\sqrt{x}-x\right)dx+\int_1^4\left(x-\sqrt{x}\right)dx$

$\left[\frac{2x^{3/2}}{3}-\frac{x^2}{2}\right]_0^1+\left[\frac{x^2}{2}-\frac{2x^{3/2}}{3}\right]_1^4$

$\frac{2}{3}-\frac{1}{2}+8-\frac{16}{3}-\left(\frac{1}{2}-\frac{2}{3}\right)=3$

9. $\int_{2}^{3}\frac{x}{\left(x^2-3\right)^2}dx$

$\int_2^3\frac{x}{\left(x^2-3\right)^2}dx=\frac{5}{12}$

$u=x^2-3, du=2x\,dx$

$\frac{1}{2}\int_2^3 2x\left(x^2-3\right)^{-2}dx$

$x=2,u=1 \mid x=3,u=6$

$\frac{1}{2}\int_1^6 u^{-2}du=\frac{1}{2}\left(\frac{-1}{u}\right)\Bigg|_1^6=\frac{-1}{12}+\frac{1}{2}=\frac{5}{12}$

10. $\int_{0}^{4}\frac{dt}{\sqrt{2t+1}}$

$u=2t+1, du=2\,dt$

$\frac{1}{2}\int_0^4 2(2t+1)^{-1/2}dt$

$t=0,u=1 \mid t=4,u=9$

$\frac{1}{2}\int_1^9 u^{-1/2}du=\left(u^{1/2}\right)\Big|_1^9=3-1-2$

11. $\int_{0}^{\pi/2}\cos^3 t\sin t\,dt$

$u=\cos t, du=-\sin t\,dt$

$-\int_0^{\pi/2}-\cos^3 t\sin t\,dt$

$t=0,u=1 \mid t=\pi/2,u=0$

$-\int_1^0 u^3du=\left(\frac{u^4}{4}\right)\Bigg|_0^1=\frac{1}{4}$

12. $\int_{0}^{\sqrt{\pi/2}}t\sin\left(\pi-t^2\right)dt$

$u=\pi-t^2, du=-2t\,dt$

$-\frac{1}{2}\int_0^{\sqrt{\pi/2}}2t\sin\left(\pi-t^2\right)dt$

$t=0,u=\pi \mid t=\sqrt{\pi/2}/2,u=\pi/2$

$-\frac{1}{2}\int_\pi^{\pi/2}\sin u\,du=\frac{1}{2}(\cos u)\Big|_\pi^{\pi/2}=\frac{1}{2}(0+1)=\frac{1}{2}$

13. $\int_0^{\pi/4} \sqrt{\tan x}\,\sec^2 x\,dx$

$u = \tan x, du = \sec^2 x\,dx$
$x = 0, u = 0 \mid x = \pi/4, u = 1$
$\int_0^1 u^{1/2} du = \left(\frac{2u^{3/2}}{3}\right)\Big|_0^1 = \frac{2}{3}$

14. $\int_0^{\pi/3} \cos x\sqrt{1-\cos^2 x}\,dx$

$\int_0^{\pi/3} \cos x \sin x\,dx$
$u = \sin x, du = \cos x\,dx$
$x = 0, u = 0 \mid x = \pi/3, u = \sqrt{3}/2$
$\int_0^{\sqrt{3}/2} u\,du = \left(\frac{u^2}{2}\right)\Big|_0^{\sqrt{3}/2} = \frac{3}{8}$

15. $\int_0^1 x\sqrt{ax^2+b}\,dx$

$u = ax^2 + b, du = 2ax\,dx$
$\frac{1}{2a}\int_0^1 2ax\sqrt{ax^2+b}\,dx$
$x = 0, u = b \mid x = 1, u = a+b$
$\frac{1}{2a}\int_b^{a+b} u^{1/2} du = \left(\frac{1}{2a}\right)\left(\frac{2u^{3/2}}{3}\right)\Big|_b^{a+b} = \frac{(ax^2+b)^{3/2} - b^{3/2}}{3a}$

16. $\int_{\pi^2/4}^{\pi^2} \frac{\sin\sqrt{x}}{\sqrt{x}}\,dx$

$u = \sqrt{x}, du = \frac{1}{2\sqrt{x}}\,dx$
$2\int_{\pi^2/4}^{\pi^2} \frac{\sin\sqrt{x}}{2\sqrt{x}}\,dx$
$x = \pi^2/4, u = \pi/2 \mid x = \pi^2, u = \pi$
$2\int_{\pi/2}^{\pi} \sin u\,du = (-2\cos u)\Big|_{\pi/2}^{\pi} = 2$

17. $\int_0^4 \left|9 - x^2\right| dx$

$9 - x^2 = 0 \Rightarrow x = \pm 3$
$\int_0^3 (9 - x^2)\,dx + \int_3^4 (x^2 - 9)\,dx$
$\left[9x - \frac{x^3}{3}\right]_0^3 + \left[\frac{x^3}{3} - 9x\right]_3^4$
$27 - 9 + \frac{64}{3} - 36 - (9 - 27) = \frac{64}{3}$

18. $\int_{-4}^4 \frac{1}{x^2}\,dx$ (Be careful!)

Since $\frac{1}{x^2}$ is not continuous at $x = 0$, the definite integral which represents area under the curve makes no sense.

If $\int_0^2 f(x)\,dx = \frac{11}{3}$ and $\int_0^6 f(x)\,dx = 15$, $f(x)$ is an even function (symmetric to the y - axis), find the following:

19. $\int_{-2}^0 f(x)\,dx = \frac{11}{3}$ 20. $\int_{-2}^2 f(x)\,dx = \frac{22}{3}$ 21. $\int_0^2 -f(x)\,dx = \frac{-11}{3}$ 22. $\int_{-2}^0 3f(x)\,dx = 11$ 23. $\int_0^2 f(3x)\,dx = 5$

If $\int_0^2 f(x)\,dx = \frac{11}{3}$ and $\int_0^6 f(x)\,dx = 15$, $f(x)$ is an odd function (symmetric to the origin), find the following:

24. $\int_{-2}^0 f(x)\,dx = \frac{-11}{3}$ 25. $\int_{-2}^2 f(x)\,dx = 0$ 26. $\int_0^2 -f(x)\,dx = \frac{-11}{3}$ 27. $\int_{-2}^0 3f(x)\,dx = -11$ 28. $\int_{-2}^2 f(3x)\,dx = 0$

23. $u = 3x \quad du = 3dx \quad \frac{1}{3}\int_0^2 3f(3x)\,dx$
$x = 0, u = 0 \mid x = 2, u = 6$
$\frac{1}{3}\int_0^6 f(u)\,du = \frac{1}{3}(15) = 5$

28. $u = 3x \quad du = 3dx \quad \frac{1}{3}\int_{-2}^2 3f(3x)\,dx$
$x = -2, u = -6 \mid x = 2, u = 6$
$\frac{1}{3}\int_{-6}^6 f(u)\,du = \frac{1}{3}(0) = 0$