Quotient Rule

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If f and g are two differentiable functions and $g(x) \neq 0$ for any x, then

$$\left(\frac{f(x)}{g(x)}\right)' = \frac{g(x)f'(x) - f(x)g'(x)}{[g(x)]^2}$$

Let us begin with the definition of a derivative.

$$\left(\frac{f(x)}{g(x)}\right)' = \lim_{h\to 0}\frac{\frac{f(x+h)}{g(x+h)} - \frac{f(x)}{g(x)}}{h}$$

$$= \lim_{h\to 0}\frac{\frac{f(x+h)g(x) - f(x)g(x+h)}{g(x+h)g(x)}}{\frac{h}{1}}$$

$$= \lim_{h\to 0}\frac{f(x+h)g(x) - f(x)g(x+h)}{h\cdot g(x+h)g(x)}$$

A common denominator was found for the numerator and then the h was flipped and brought to the bottom. (Recall when two fractions are divided, multiply by the reciprocal.)

Now, factor out $g(x+h)g(x)$ from the denominator.

$$\left(\frac{f(x)}{g(x)}\right)' = \lim_{h\to 0}\frac{1}{g(x+h)g(x)}\cdot\frac{f(x+h)g(x) - f(x)g(x+h)}{h}$$

Now, as with the proof of the product rule, we have to do some magic. Subtract and add $f(x)g(x)$ to the numerator and denominator. Doing this yields:

$$\left(\frac{f(x)}{g(x)}\right)' = \lim_{h\to 0}\frac{1}{g(x+h)g(x)}\cdot\frac{f(x+h)g(x) - f(x)g(x) + f(x)g(x) - f(x)g(x+h)}{h}$$

Recall that the limit of a product is a product of the limits. Using this fact, we get:

$$\left(\frac{f(x)}{g(x)}\right)' = \lim_{h\to 0}\frac{1}{g(x+h)g(x)}\cdot\lim_{h\to 0}\frac{f(x+h)g(x) - f(x)g(x) + f(x)g(x) - f(x)g(x+h)}{h}.$$

Now, break up the second limit into two pieces because the limit of a sum is the sum of the limits.

$$\left(\frac{f(x)}{g(x)}\right)' = \lim_{h\to 0}\frac{1}{g(x+h)g(x)}\cdot\left[\lim_{h\to 0}\frac{f(x+h)g(x)-f(x)g(x)}{h}+\lim_{h\to 0}\frac{f(x)g(x)-f(x)g(x+h)}{h}\right]$$

Factoring out a *g*(*x*) from the second limit and a -*f*(*x*) from the third limit, we get:

$$\left(\frac{f(x)}{g(x)}\right)' = \lim_{h\to 0}\frac{1}{g(x+h)g(x)}\cdot\left[\lim_{h\to 0}\frac{g(x)\left[f(x+h)-f(x)\right]}{h}-\lim_{h\to 0}\frac{f(x)\left[g(x+h)-g(x)\right]}{h}\right]$$

Now, by the property of limits, we have the following are true:

$$\begin{aligned}\lim_{h\to 0}\frac{1}{g(x+h)g(x)} &= \lim_{h\to 0}\frac{1}{g(x+h)}\cdot\lim_{h\to 0}\frac{1}{g(x)}\\ &= \frac{1}{g(x)}\cdot\frac{1}{g(x)}\\ &= \frac{1}{\left[g(x)\right]^2}\end{aligned}$$

$$\begin{aligned}\lim_{h\to 0}\frac{g(x)\left[f(x+h)-f(x)\right]}{h} &= \lim_{h\to 0}g(x)\cdot\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}\\ &= g(x)f'(x)\end{aligned}$$

$$\begin{aligned}\lim_{h\to 0}\frac{f(x)\left[g(x+h)-g(x)\right]}{h} &= \lim_{h\to 0}f(x)\cdot\lim_{h\to 0}\frac{g(x+h)-g(x)}{h}\\ &= f(x)g'(x)\end{aligned}$$

Finally, putting this all together, we have:

$$\begin{aligned}\left(\frac{f(x)}{g(x)}\right)' &= \frac{1}{\left[g(x)\right]^2}\cdot\left[g(x)f'(x)-f(x)g'(x)\right]\\ &= \frac{g(x)f'(x)-f(x)g'(x)}{\left[g(x)\right]^2}\end{aligned}$$

This is what we wanted to show, and so we are done.