Product Rule

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If f and g are two differentiable functions, then

$$\left(f(x)g(x)\right)' = f(x)g'(x) + g(x)f'(x)$$

Let us begin with the definition of a derivative.

$$\left(f(x)g(x)\right)' = \lim_{h\to 0}\frac{f(x+h)g(x+h) - f(x)g(x)}{h}$$

The key is to subtract and add a term: $f(x+h)g(x)$. You need to know to do this to make any progress. Doing this, we get the following:

$$\left(f(x)g(x)\right)' = \lim_{h\to 0}\frac{f(x+h)g(x+h) - f(x+h)g(x) + f(x+h)g(x) - f(x)g(x)}{h}.$$

From the property of limits, we can break the limit into two pieces because the limit of a sum is the sum of the limits. And so, we have:

$$\left(f(x)g(x)\right)' = \lim_{h\to 0}\frac{f(x+h)g(x+h) - f(x+h)g(x)}{h} + \lim_{h\to 0}\frac{f(x+h)g(x) - f(x)g(x)}{h}$$

Factoring a $f(x+h)$ on the first limit and $g(x)$ from the second limit we get:

$$\left(f(x)g(x)\right)' = \lim_{h\to 0}\frac{f(x+h)\left[g(x+h) - g(x)\right]}{h} + \lim_{h\to 0}\frac{g(x)\left[f(x+h) - f(x)\right]}{h}$$

Another property of limits says that the limit of a product is a product of the limits. Using this fact, we can rewrite the limit as:

$$\left(f(x)g(x)\right)' = \lim_{h\to 0} f(x+h)\lim_{h\to 0}\frac{g(x+h) - g(x)}{h} + \lim_{h\to 0} g(x)\lim_{h\to 0}\frac{f(x+h) - f(x)}{h}$$

By definition, though, $\lim_{h\to 0}\frac{f(x+h) - f(x)}{h} = f'(x)$ and $\lim_{h\to 0}\frac{g(x+h) - g(x)}{h} = g'(x)$. Also, $\lim_{h\to 0} f(x+h) = f(x)$ and $\lim_{h\to 0} g(x) = g(x)$, since they do not depend on *h*.

So, we have $\left(f(x)g(x)\right)' = f(x)g'(x) + g(x)f'(x)$, which is what we wanted to show.